The role of the Seiberg-Witten field redefinition in renormalization of noncommutative chiral electrodynamics

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Abstract

It has been conjectured in the literature that renormalizability of the θ-expanded noncommutative gauge theories improves when one takes into account full nonuniqueness of the Seiberg-Witten expansion, which relates noncommutative ('high-energy') with commutative ('low-energy') fields. In order to check this conjecture we analyze renormalizability of the noncommutative chiral electrodynamics: we quantize the action which contains all possible terms implied by the SW map. After renormalization we arrive at a different theory in which the relation between the coupling constants is changed. This means that the θ-expanded chiral electrodynamics is not renormalizable: when fermions are included, the SW expansion is not preserved in quantization.

I.   INTRODUCTION

Basic noncommutativity of spacetime coordinates [1] is a very plausible idea when one ponders two singularity problems of classical and quantum field theory: singular solutions and renormalizability. This can be easily seen: when coordinates are represented by operators $\hat{x}^{\mu}$, commutation relations

$$[\hat{x}^{\mu}, \hat{x}^{\nu}] = i\hat{\theta}^{\mu\nu}(\hat{x}), \qquad \mu, \nu = 1, \ldots, d \tag{1}$$

put a lower bound of order $|\theta|^{\frac{1}{2}}$ on coordinate measurements and an upper bound of order $\hbar|\theta|^{-\frac{1}{2}}$ on momentum measurements. Here $|\theta|^{\frac{1}{2}}$ is the scale of noncommutativity $\hat{\theta}^{\mu\nu}$. This property is desirable as a cure to both of the mentioned problems, but it opens a whole range of questions, conceptual and concrete. First, one has to define operationally a 'noncommutative space' $\mathcal{A}$ which fulfills (1), preferrably with a notion of smoothness or differentiability, and second, fields on it. This for itself is a difficult mathematical problem. But a necessary constraint which one wishes to impose is that noncommutative field theories have good commutative limit, the one which is experimentally well established at the present length scale, providing at the same time resolution to the singularity problems at the noncommutativity scale.

This of course is not an easy task. The most feasible model of noncommutativity which we usually start with is the space with constant, 'canonical' noncommutativity:

$$[\hat{x}^{\mu}, \hat{x}^{\nu}] = i\theta^{\mu\nu} = \text{const}, \tag{2}$$

because fields $\hat{\phi}$, $\hat{\chi}$ on it can be represented by functions on ordinary $\mathbb{R}^4$. The field multiplication is given by the Moyal-Weyl star-product:

$$\hat{\phi}(x) \star \hat{\chi}(x) = e^{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^{\mu}}\frac{\partial}{\partial y^{\nu}}} \hat{\phi}(x)\hat{\chi}(y)|_{y \to x}. \tag{3}$$

This representation is called the Moyal space. The Moyal space is a flat noncomutative space, but clearly in any number of dimensions except in $d = 2$, constant commutator (2) breaks the Lorentz symmetry.

Gauge symmetries on the Moyal space can be, in principle, introduced in a straightforward way. For example, for spinor field $\hat{\psi}$ one can define the action of the noncommutative U(1) gauge group by

$$\hat{\psi}' = \hat{U}\hat{\psi}, \tag{4}$$

the $\hat{U}$ are unitary elements of $\mathcal{A}$. As coordinates $\hat{x}^{\mu}$ generate $\mathcal{A}$, $\hat{U}$ are always expressible as functions, $\hat{U} = \hat{U}(\hat{x}^{\mu})$: we are dealing with local symmetry. The group action can also be the adjoint, $\hat{\psi}' = \hat{U}\hat{\psi}\hat{U}^{-1}$ or the right action, $\hat{\psi}' = \hat{\psi}\hat{U}^{-1}$. Obviously, the noncommutative U(1) group is not abelian, and therefore transformation properties of the gauge potential resemble those of nonabelian theories. In fact, the expansion of the potential in terms of the Lie algebra generators, $\hat{A}_{\mu} = \hat{A}_{\mu}^{a}T^{a}$, is possible only for the U(N) groups. When one considers general noncommutative spaces, quite often one can define only infinitesimal symmetry transformations.

In other aspects, also, noncommutative gauge symmetries exibit new features. They are particularly interesting when one considers noncommutative spaces defined as limits of $N \times N$ matrix spaces for $N \to \infty$. Then for example,

the elements of the noncommutative U(1) gauge group, the unitary $N \times N$ matrices, belong at the same time to the ordinary U(N): in a way, the local noncommutative U(1) can be identified with the U(∞) on the commutative space. This connection can be extended to the Chern-Simons action, [2]. The natural mixing of gauge and spatial degrees of freedom in noncommutative geometry [3], can be further elaborated in the framework of matrix models as a possibility to interpret gravity as emergent, that is, as the U(1) part of the U(N) symmetry group, [4]. These novel aspects and relations should be investigated and understood in more details.

In this paper we will be mainly concerned with gauge theories which include fermions. We said that one of the important constraints on noncommutative theories is their commutative limit $\theta \to 0$. In this limit one naturally expects that noncommutative fields $\hat{A}_\mu$, $\hat{\psi}$ reduce to commutative gauge and matter fields A_μ and ψ,

$$\hat{A}_\mu|_{\theta=0} = A_\mu, \qquad \hat{\psi}|_{\theta=0} = \psi, \tag{5}$$

as for $\theta = 0$ the star product reduces to the ordinary one. We see therefore that all noncommutative theories have the same commutative limit. On the other hand, starting from a specific commutative theory one can get various deformations: noncommutative generalization is not unique. But the noncommutative structure of the space itself can give some restrictions which reduce the number of possible models.

We mentioned that for symmetry transformations defined by (4) on the Moyal space, only the U(N) groups can be consistently represented. Various aspects of this kind of models were analyzed in the literature [5–9], and it was shown that in perturbative quantization they behave worse than commutative gauge theories: the ultraviolet divergences 'propagate' to the infrared sector (UV/IR mixing). Another widely explored possibility of representing gauge symmetries is the enveloping algebra formalism, in which one enlarges the Lie algebra of the group to the enveloping algebra, expanding the gauge potential in the symmetrized products of the group generators T^a,

$$\hat{A}_\mu = \hat{A}_\mu^a T^a + \hat{A}_\mu^{ab}\{T^a, T^b\} + \dots \tag{6}$$

In this approach there is no restriction on the type of the gauge group, but the UV/IR mixing remains in straightforward quantization, [10, 11]. In the original version of the enveloping-algebra formalism [12–14], one expands the theory in θ: the expansion of fields is called the Seiberg-Witten (SW) map [15]. This expansion, apart from giving the effective low-energy theory and the new interactions, can in principle be used to define the quantization procedure. The idea is the following: one first quantizes the theory in the first order, then proceeds to the second and higher orders by using some kind of iterative procedure. Higher orders of fields in the SW expansion are related to lower orders by gauge symmetries [16, 17], so one can hope that renormalizability will be achieved through a noncommutative version of the Ward identities. In addition, the SW expansion has an amount of nonuniqueness which increases the number of possible counterterms for renormalization.

Therefore, in order to discuss the θ-expanded theories we must first investigate their behavior in linear order. The results, especially for the pure gauge theories, are quite encouraging. The idea that the SW nonuniqueness [18] can be used to obtain renormalizability was proposed and used first in [19] to show that the photon self-energy diagram in the noncommutative U(1) theory is renormalizable to all orderds in θ. Linear-order action for the SU(N) gauge theory was analyzed in [20, 21] and found to be renormalizable; a similar result was obtained for the gauge sector

of a suitably defined noncommutative Standard Model, [22]. Inclusion of the matter on the other hand presents a difficulty. For some time it was believed that fermions cannot be successfully incorporated into a renormalizable theory because of the 4ψ divergence, [23, 24]; however, this divergence is absent in chiral theories, [25]. In fact, we showed that all perturbative divergences of noncommutative chiral electrodynamics are potentially removable by the Seiberg-Witten redefinition of fields, [26]. Similar positive results were obtained in [27–29] for the GUT inspired anomaly-safe models with chiral fermions: it was found that all linear-order divergences are given by marginal operators. A counterargument was given by Armoni [30] who, comparing the expanded to the unexpanded theories, argues that the sum of marginal operators of different orders gives in fact a relevant operator which prevents renormalizability. To clarify the question of renormalizability and decide whether the potential renormalizability of the θ-expanded chiral electrodynamics obtained in [26] is in fact actual, we undertake in this paper a task to renormalize the model explicitly. In order to do that we have to take into account the full nonuniqueness of the SW expansion, which gives six new interaction terms in the action. The new coupling constants κ_i however are constrained by one relation, (32). We calculate all one-loop divergences and perform renormalization of κ_i, including noncommutativity $\theta^{\mu\nu}$: we obtain that renormalization of the coupling constants violates the constraint equation. Our conclusion is: the SW expansion cannot hold simultaneously for the bare and the dressed fields. This means that the model defined as low-energy part of a basic noncommutative gauge theory given by (8) and (9), is not renormalizable.

II.   NONCOMMUTATIVE CHIRAL ELECTRODYNAMICS

Let us introduce the lagrangian. We wish to discuss minimal noncommutative extension of the commutative chiral electrodynamics defined by the action

$$S_{\mathrm{C}} = \int \mathrm{d}^4x \left(\mathrm{i}\bar{\varphi}\bar{\sigma}^\mu(\partial_\mu + \mathrm{i}qA_\mu)\varphi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \right), \tag{7}$$

where φ is a left chiral fermion of charge q, A_μ is the vector potential and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength. Noncommutative fields $\hat{\varphi}$, $\hat{A}_\mu$ and $\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu + \mathrm{i}q[\hat{A}_\mu \overset{\star}{,} \hat{A}_\nu]$ are also represented by functions of commutative coordinates; the corresponding noncommutative action is given by

$$S_{\mathrm{NC}} = \int \mathrm{d}^4x \left(\mathrm{i}\hat{\bar{\varphi}} \star \bar{\sigma}^\mu(\partial_\mu + \mathrm{i}q\hat{A}_\mu) \star \hat{\varphi} - \frac{1}{4}\hat{F}_{\mu\nu} \star \hat{F}^{\mu\nu} \right). \tag{8}$$

Two sets of fields are related by the Seiberg-Witten map:

$$\hat{A}_\rho = A_\rho + \sum_{n=1} A_\rho^{(n)}, \qquad \hat{\varphi} = \varphi + \sum_{n=1} \varphi^{(n)}. \tag{9}$$

This map is an expansion in powers of noncommutativity $\theta^{\mu\nu}$ which is designated by index (n): in the commutative limit higher powers vanish and the initial values of fields are $\varphi^{(0)} = \varphi$, $A_\rho^{(0)} = A_\rho$. Seiberg-Witten expansion (9) can be seen as solution to the condition that infinitesimal symmetry transformations close. The simplest solution in

linear order is given by [12–14]:

$$\hat{A}_\rho = A_\rho + \frac{1}{4}q\,\theta^{\mu\nu}\{A_\mu, \partial_\nu A_\rho + F_{\nu\rho}\}, \tag{10}$$

$$\hat{F}_{\rho\sigma} = F_{\rho\sigma} - \frac{1}{2}q\,\theta^{\mu\nu}\{F_{\mu\rho}, F_{\nu\sigma}\}$$

$$+ \frac{1}{4}q\,\theta^{\mu\nu}\{A_\mu, (\partial_\nu + D_\nu)F_{\rho\sigma}\}, \tag{11}$$

$$\hat{\varphi} = \varphi + \frac{1}{2}q\,\theta^{\mu\nu}A_\mu\partial_\nu\varphi, \tag{12}$$

where D_μ denotes the commutative covariant derivative, $D_\mu\varphi = (\partial_\mu + iqA_\mu)\varphi$. It is possible to generate higher orders in the expansion from linear order, [16]. In addition, whenever we have a particular solution $A_\rho^{(n)}$, $\varphi^{(n)}$, we can obtain a more general one by adding arbitrary gauge covariant expressions $\mathbf{A}_\rho^{(n)}$, $\mathbf{\Phi}^{(n)}$ of the same order [18, 19]:

$$A_\rho'^{(n)} = A_\rho^{(n)} + \mathbf{A}_\rho^{(n)}, \qquad \varphi'^{(n)} = \varphi^{(n)} + \mathbf{\Phi}^{(n)}. \tag{13}$$

This property is called the Seiberg-Witten nonuniqueness: it means that the relation between the 'physical', high-energy fields $\hat{A}_\rho$, $\hat{\varphi}$, and the usual, experimentally observed low-energy fields A_ρ, φ is not uniquely defined beyond the zeroth order in θ. One intuitively expects that such a difference would be unobservable. But in fact the SW field redefinition can change not only the dispersion relations or the cross sections: it changes even the renormalization properties of the theory, and this happens when fermionic matter is included. Therefore, renormalizability of the theory was proposed in the literature as a criterion which fixes the nonuniqueness of the Seiberg-Witten expansion.

We shall discuss θ-linear order of the chiral electrodynamics. Using SW expansions (10-12) we obtain the action

$$\mathcal{L}_{\mathrm{NC}} = \mathcal{L}_{0,A} + \mathcal{L}_{0,\varphi} + \mathcal{L}_{1,A} + \mathcal{L}_2, \tag{14}$$

with

$$\mathcal{L}_{0,A} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{15}$$

$$\mathcal{L}_{0,\varphi} = i\bar{\varphi}\bar{\sigma}^\mu(D_\mu\varphi), \tag{16}$$

$$\mathcal{L}_{1,A} = \frac{1}{2}q\,\theta^{\mu\nu}\left(F_{\mu\rho}F_{\nu\sigma}F^{\rho\sigma} - \frac{1}{4}F_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right), \tag{17}$$

$$\mathcal{L}_2 = -\frac{i}{16}q\,\theta^{\mu\nu}\Delta_{\mu\nu\rho}^{\alpha\beta\gamma}F_{\alpha\beta}\,\bar{\varphi}\,\bar{\sigma}^\rho(D_\gamma\varphi) + \text{h.c.}; \tag{18}$$

we denote in the following $\varepsilon^{\alpha\beta\gamma\delta}\varepsilon_{\mu\nu\rho\delta} = -\Delta_{\mu\nu\rho}^{\alpha\beta\gamma}$. Action (14) was analyzed in [26] and it was shown that, as it stands, it is not renormalizable. However, all divergences are of the form implied by a SW redefinition and therefore we conjectured that there exists another expansion within allowed class (13) which gives a renormalizable theory.

In order to check this conjecture, we need to expand the lagrangian using the most general first-order SW solution. Therefore instead of (10-12) we use

$$A_\rho' = A_\rho + \mathbf{A}_\rho^{(1)}, \qquad \varphi' = \varphi + \mathbf{\Phi}^{(1)}, \tag{19}$$

where $\mathbf{A}_\rho^{(1)}$ and $\mathbf{\Phi}^{(1)}$ are covariant expressions of first order in θ:

$$
\begin{aligned}
\mathbf{A}_\rho^{(1)} &= a_1 \theta^{\mu\nu} \varepsilon_{\mu\rho\sigma\tau} (\partial_\nu F^{\sigma\tau}) + a_2 \theta^{\mu\nu} \varepsilon_{\mu\nu\rho\tau} (\partial_\sigma F^{\tau\sigma}) \\
&\quad + a_3 \theta^{\mu\nu} \varepsilon_{\mu\nu\tau\sigma} (\partial_\rho F^{\tau\sigma}),
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
\mathbf{\Phi}^{(1)} &= \mathrm{i} b_1 \theta^{\mu\nu} \sigma_{\mu\nu} (D^2 \varphi) + b_2 q \theta^{\mu\nu} F_\mu{}^\rho \sigma_{\nu\rho} \varphi \\
&\quad + b_3 q \theta^{\mu\nu} F_{\mu\nu} \varphi + \mathrm{i} b_4 q \theta^{\mu\nu} \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} \varphi,
\end{aligned}
\tag{21}
$$

and constants a_i and b_i are real. This changes the initial action to

$$
S'_{\mathrm{NC}} = S_{\mathrm{NC}} + \Delta S_{\mathrm{SW}}^{(1)},
\tag{22}
$$

with

$$
\begin{aligned}
\Delta S_{\mathrm{SW}}^{(1)} &= \int \mathrm{d}^4 x \left((D_\rho F^{\rho\mu}) \mathbf{A}_\mu^{(1)} - q \bar{\varphi} \bar{\sigma}^\mu \varphi \mathbf{A}_\mu^{(1)} \right. \\
&\quad \left. + \left(\mathrm{i} \bar{\varphi} \bar{\sigma}^\mu (D_\mu \mathbf{\Phi}^{(1)}) + \mathrm{h.c.} \right) \right).
\end{aligned}
\tag{23}
$$

When we introduce (20-21) and simplify the action using various identities, we obtain

$$
\begin{aligned}
\Delta \mathcal{L}_{\mathrm{SW}}^{(1)} &= \mathrm{i} \frac{b_1}{2} \theta^{\mu\nu} \varepsilon_{\mu\nu\rho\sigma} \bar{\varphi} \bar{\sigma}^\sigma D^\rho D^2 \varphi \\
&\quad + q \theta^{\mu\nu} \left[-\mathrm{i} \left(b_1 + \frac{b_2}{2} \right) F_{\mu\rho} \bar{\varphi} \bar{\sigma}_\nu D^\rho \varphi + \mathrm{i} \frac{b_2}{2} F_{\mu\rho} \bar{\varphi} \bar{\sigma}^\rho D_\nu \varphi + \mathrm{i} b_3 F_{\mu\nu} \bar{\varphi} \bar{\sigma}^\rho D_\rho \varphi \right. \\
&\quad \left. + \left(a_1 + a_2 - \frac{b_2}{4} \right) \varepsilon_{\mu\rho\sigma\tau} F^{\rho\sigma} \bar{\varphi} \bar{\sigma}^\tau D_\nu \varphi + \left(a_3 - \frac{a_2}{2} + b_4 \right) \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} \bar{\varphi} \bar{\sigma}^\tau D_\tau \varphi \right] + \mathrm{h.c.}.
\end{aligned}
$$

This form is in a way canonical as it contains minimal number of terms. The new lagrangian, our starting point for quantization, reads

$$
\mathcal{L}'_{\mathrm{NC}} = \mathcal{L}_{\mathrm{C}} + \mathcal{L}_{1,A} + (1 + \kappa_2) \mathcal{L}_2 + \sum_{i=3}^7 \kappa_i \mathcal{L}_i,
\tag{24}
$$

where κ_i, $i = 2, \ldots, 7$ are the coupling constants introduced as

$$
\kappa_2 = -b_2, \quad \kappa_3 = \frac{b_1}{2}, \quad \kappa_4 = -b_1 - \frac{b_2}{2},
\tag{25}
$$

$$
\kappa_5 = \frac{b_2}{4} + b_3, \quad \kappa_6 = a_1 + a_2 - \frac{b_2}{4}, \quad \kappa_7 = a_3 - \frac{a_2}{2} + b_4,
\tag{26}
$$

and

$$
\mathcal{L}_3 = \mathrm{i} \theta^{\mu\nu} \varepsilon_{\mu\nu\rho\sigma} \bar{\varphi} \bar{\sigma}^\sigma D^\rho D^2 \varphi + \mathrm{h.c.}
\tag{27}
$$

$$
\mathcal{L}_4 = \mathrm{i} q \theta^{\mu\nu} F_{\mu\rho} \bar{\varphi} \bar{\sigma}_\nu D^\rho \varphi + \mathrm{h.c.}
\tag{28}
$$

$$
\mathcal{L}_5 = \mathrm{i} q \theta^{\mu\nu} F_{\mu\nu} \bar{\varphi} \bar{\sigma}^\rho D_\rho \varphi + \mathrm{h.c.}
\tag{29}
$$

$$
\mathcal{L}_6 = q \theta^{\mu\nu} \varepsilon_{\mu\rho\sigma\tau} F^{\rho\sigma} \bar{\varphi} \bar{\sigma}^\tau D_\nu \varphi + \mathrm{h.c.}
\tag{30}
$$

$$
\mathcal{L}_7 = q \theta^{\mu\nu} \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} \bar{\varphi} \bar{\sigma}^\tau D_\tau \varphi + \mathrm{h.c.}.
\tag{31}
$$

Note that not all coupling constants are independent: there is a relation between them,

$$\kappa_2 - 4\kappa_3 - 2\kappa_4 = 0. \tag{32}$$

We shall see that this relation is broken in quantization.

III.   QUANTIZATION

For quantization we use the background field method. The procedure for this kind of a model was developed in details in [26] so we will not repeat it here. The main difference is that now, instead of one, we have six fermion-photon vertices; in addition, the fermion propagator has noncommutative correction which we also treat perturbatively. For the purpose of calculation of functional integrals we introduce the Majorana spinor ψ instead of the chiral φ, $\psi = \begin{pmatrix} \varphi_\alpha \\ \bar{\varphi}^{\dot{\alpha}} \end{pmatrix}$. Rewriting the action in Majorana spinors, for the commutative part of the spinor lagrangian we obtain $\mathcal{L}_{0,\varphi} = \frac{i}{2} \bar{\psi} \gamma^\mu (\partial_\mu - iq\gamma_5 A_\mu)\psi$; noncommutative terms are expressed likewise.

The one-loop effective action is given by expansion

$$
\begin{aligned}
\Gamma^{(1)} &= \frac{i}{2} \text{STr} \log \left(\mathcal{I} + \Box^{-1} N_1 + \Box^{-1} T_0 + \Box^{-1} T_1 + \Box^{-1} T_2 \right) \\
&= \frac{i}{2} \sum \frac{(-1)^{n+1}}{n} \text{STr} \left(\Box^{-1} N_1 + \Box^{-1} T_0 + \Box^{-1} T_1 + \Box^{-1} T_2 \right)^n .
\end{aligned}
\tag{33}
$$

In this formula, N_1 denotes a matrix which is obtained from commutative 3-vertices after the expansion of quantum fields around the stationary classical configuration. It is given by

$$N_1 = q \begin{pmatrix} 0 & i\bar{\psi}\gamma_5\gamma^\kappa\slashed{\partial} \\ -\gamma_5\gamma^\lambda\psi & i\gamma_5\slashed{A}\slashed{\partial} \end{pmatrix}.$$

The T_0, T_1 and T_2 are defined in analogy but related to the terms linear in θ: T_0 corresponds to the 2-vertex, that is, to the noncommutative correction of the fermion propagator, while T_1 and T_2 are obtained from 3- and 4-vertices. The expansion gives

$$T_0 = 2\kappa_3 \theta^{\mu\nu} \varepsilon_{\mu\nu\rho\sigma} \begin{pmatrix} 0 & 0 \\ 0 & \gamma^\sigma \slashed{\partial}\partial^\rho\Box \end{pmatrix}.$$

T_1 is the sum of six terms, $T_1 = \sum_{i=2}^{7} T_1^{\kappa_i}$, with

$$T_1^{\kappa_2} \;=\; \frac{1}{8}\kappa_2 q\theta^{\mu\nu}\Delta_{\mu\nu\rho}^{\alpha\beta\gamma}\begin{pmatrix} 0 & 2\delta_\alpha^\kappa(\partial_\beta\bar\psi)\gamma^\rho\partial_\gamma\slashed\partial \\ 2\mathrm{i}\delta_\alpha^\lambda\gamma^\rho(\partial_\beta\psi)\partial_\gamma & F_{\alpha\beta}\gamma^\rho\partial_\gamma\slashed\partial \end{pmatrix},$$

$$T_1^{\kappa_3} \;=\; 2\kappa_3 q\theta^{\mu\nu}\varepsilon_{\mu\nu\rho\sigma}\begin{pmatrix} 0 & \mathrm{i}(g^{\rho\kappa}\bar\psi\Box - (\partial^\rho\bar\psi)\partial^\kappa + \bar\psi\partial^\rho\partial^\kappa)\gamma^5\gamma^\sigma\slashed\partial \\ \gamma^5\gamma^\sigma(g^{\rho\lambda}(\Box\psi) - \overleftarrow{\partial}{}^\rho(\partial^\lambda\psi) + (\partial^\rho\partial^\lambda\psi)) & \mathrm{i}(A^\rho\Box - \overleftarrow{\partial}{}^\rho A^\tau\partial_\tau + A_\tau\partial^\rho\partial^\tau)\gamma^5\gamma^\sigma\slashed\partial \end{pmatrix},$$

$$T_1^{\kappa_4} \;=\; 2\kappa_4 q\theta^{\mu\nu}\begin{pmatrix} 0 & (\overleftarrow{\partial}{}_\mu g_{\rho\kappa} - \overleftarrow{\partial}{}_\rho g_{\mu\kappa})\bar\psi\gamma_\nu\partial^\rho\slashed\partial \\ \mathrm{i}\gamma_\nu(\partial^\rho\psi)(g_{\rho\lambda}\partial_\mu - g_{\mu\lambda}\partial_\rho) & F_{\mu\rho}\gamma_\nu\partial^\rho\slashed\partial \end{pmatrix},$$

$$T_1^{\kappa_5} \;=\; 2\kappa_5 q\theta^{\mu\nu}\begin{pmatrix} 0 & 2\overleftarrow{\partial}{}_\mu g_{\nu\kappa}\bar\psi\Box \\ 2\mathrm{i}\gamma^\rho(\partial_\rho\psi)g_{\nu\lambda}\partial_\mu & F_{\mu\nu}\Box \end{pmatrix},$$

$$T_1^{\kappa_6} \;=\; 2\kappa_6 q\theta^{\mu\nu}\varepsilon_{\mu\rho\sigma\tau}\begin{pmatrix} 0 & 2\mathrm{i}\overleftarrow{\partial}{}^\rho\delta_\kappa^\sigma\bar\psi\gamma^\tau\gamma_5\partial_\nu\slashed\partial \\ -2\delta_\lambda^\sigma\gamma^\tau\gamma_5(\partial_\nu\psi)\partial^\rho & \mathrm{i}F^{\rho\sigma}\gamma^\tau\gamma_5\partial_\nu\slashed\partial \end{pmatrix},$$

$$T_1^{\kappa_7} \;=\; 2\kappa_7 q\theta^{\mu\nu}\varepsilon_{\mu\nu\rho\sigma}\begin{pmatrix} 0 & -2\mathrm{i}\overleftarrow{\partial}{}^\rho\delta_\kappa^\sigma\bar\psi\gamma^5\Box \\ -2\delta_\lambda^\sigma\gamma^\tau\gamma_5(\partial_\tau\psi)\partial^\rho & -\mathrm{i}F^{\rho\sigma}\gamma_5\Box \end{pmatrix},$$

and from 4-vertices we obtain that $T_2 = \sum_{i=2}^{5}T_2^{\kappa_i}$ with

$$T_2^{\kappa_2} \;=\; \frac{1}{8}\kappa_2 q^2\theta^{\mu\nu}\Delta_{\mu\nu\rho}^{\alpha\beta\gamma}\begin{pmatrix} 2\delta_\beta^\lambda\delta_\gamma^\kappa\bar\psi\gamma^\rho\gamma_5\psi\partial_\alpha & -\mathrm{i}\delta_\gamma^\kappa F_{\alpha\beta}\bar\psi\gamma^\rho\gamma^5\slashed\partial - 2\mathrm{i}\overleftarrow{\partial}{}_\alpha\delta_\beta^\kappa A_\gamma\bar\psi\gamma^\rho\gamma^5\slashed\partial \\ F_{\alpha\beta}\delta_\gamma^\lambda\gamma^\rho\gamma^5\psi + 2A_\gamma\delta_\beta^\lambda\gamma^\rho\gamma^5\psi\partial_\alpha & -\mathrm{i}F_{\alpha\beta}A_\gamma\gamma^\rho\gamma_5\slashed\partial \end{pmatrix},$$

$$T_2^{\kappa_3} \;=\; 2\kappa_3 q^2\theta^{\mu\nu}\varepsilon_{\mu\nu\rho\sigma}\begin{pmatrix} -\mathrm{i}(2\delta_\lambda^\rho\bar\psi\gamma^\sigma(\partial_\kappa\psi) + g_{\lambda\kappa}\bar\psi\gamma^\sigma(\partial^\rho\psi)) & -2(A^\rho\bar\psi\partial_\kappa + A^\tau\delta_\kappa^\rho\bar\psi\partial_\tau + A_\kappa\bar\psi\partial^\rho)\gamma^\sigma\slashed\partial \\ -2\mathrm{i}\gamma^\sigma(A^\rho(\partial_\lambda\psi) + \delta_\lambda^\rho A^\tau(\partial_\tau\psi) + A_\lambda(\partial^\rho\psi)) & -\gamma^\sigma(2A^\tau A^\rho\partial_\tau + A^\tau A_\tau\partial^\rho)\slashed\partial \end{pmatrix},$$

$$T_2^{\kappa_4} \;=\; 2\kappa_4 q^2\theta^{\mu\nu}\begin{pmatrix} -g^{\rho\kappa}\bar\psi\gamma_5\gamma_\nu\psi(\delta_\rho^\lambda\partial_\mu - \delta_\mu^\lambda\partial_\rho) & \mathrm{i}(F_{\mu\kappa} + (\overleftarrow{\partial}{}_\mu g_{\kappa\rho} - \overleftarrow{\partial}{}_\rho g_{\kappa\mu})A^\rho)\bar\psi\gamma_5\gamma_\nu\slashed\partial \\ -\gamma_5\gamma_\nu\psi(F_{\mu\lambda} + A^\rho(g_{\lambda\rho}\partial_\mu - g_{\lambda\mu}\partial_\rho)) & \mathrm{i}F_{\mu\rho}A^\rho\gamma_5\gamma_\nu\slashed\partial \end{pmatrix},$$

$$T_2^{\kappa_5} \;=\; 2\kappa_5 q^2\theta^{\mu\nu}\begin{pmatrix} 2\delta_\mu^\lambda\bar\psi\gamma_5\gamma^\kappa\psi\partial_\nu & \mathrm{i}(F_{\mu\nu}\bar\psi\gamma_5\gamma^\kappa + 2\overleftarrow{\partial}{}_\mu\delta_\nu^\kappa A^\rho\bar\psi\gamma_5\gamma_\rho)\slashed\partial \\ -F_{\mu\nu}\gamma_5\gamma^\lambda\psi + 2A^\rho\gamma^5\gamma_\rho\psi\delta_\nu^\lambda\partial_\mu & \mathrm{i}F_{\mu\nu}\gamma_5\slashed{A}\slashed\partial \end{pmatrix}.$$

These operators are the basic ingredients for the perturbation theory.

IV. DIVERGENCES AND RENORMALIZATION

By the power counting one can fix the terms which give divergent contributions. They come from

$$\Gamma_{\text{div}}^{(1)} = \frac{i}{2} \, \text{STr} \left(-\frac{1}{2} (\Box^{-1} N_1 \Box^{-1} N_1) + \frac{1}{3} (\Box^{-1} N_1 \Box^{-1} N_1 \Box^{-1} N_1) \right. \tag{34}$$

$$\left. - (\Box^{-1} N_1 \Box^{-1} T_1) - (\Box^{-1} N_1 \Box^{-1} N_1 \Box^{-1} N_1 \Box^{-1} T_0) + (\Box^{-1} N_1 \Box^{-1} N_1 \Box^{-1} T_1) - (\Box^{-1} N_1 \Box^{-1} T_2) \right) \bigg|_{\text{div}} .$$

We have calculated divergences by dimensional regularization. In this very demanding calculation our main aid, apart from the *Mathematica* based package *MathTensor*, was the gauge covariance of the background field method. Omitting the intermediate steps, we write the final result:

$$\Gamma_{\text{div}}^{(1)} = \frac{1}{(4\pi)^2 \epsilon} q^2 \int d^4 x \left(i \bar{\psi} \gamma^\mu (\partial_\mu - i q \gamma_5 A_\mu) \psi - \frac{1}{3} F_{\mu\nu} F^{\mu\nu} \right. \tag{35}$$

$$+ \theta^{\mu\nu} \left(\frac{i}{12} \varepsilon_{\mu\nu\rho\sigma} \bar{\psi} \gamma^\sigma D^\rho D^\tau D_\tau \psi + q \left(\frac{2}{3} F_{\mu\rho} F_{\nu\sigma} F^{\rho\sigma} - \frac{1}{6} F_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} - \frac{5i}{6} F_{\mu\rho} \bar{\psi} \gamma^\rho D_\nu \psi \right. \right.$$

$$\left. + \frac{i}{6} F_{\mu\rho} \bar{\psi} \gamma_\nu D^\rho \psi + \frac{2i}{3} F_{\mu\nu} \bar{\psi} \gamma^\rho D_\rho \psi + \frac{1}{6} \varepsilon_{\mu\rho\sigma\tau} F^{\rho\sigma} \bar{\psi} \gamma_5 \gamma^\tau D_\nu \psi - \frac{7}{8} \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} \bar{\psi} \gamma_5 \gamma^\tau D_\tau \psi \right)$$

$$+ \kappa_2 \left(\frac{i}{12} \varepsilon_{\mu\nu\rho\sigma} \bar{\psi} \gamma^\sigma D^\rho D^\tau D_\tau \psi + q \left(\frac{2}{3} F_{\mu\rho} F_{\nu\sigma} F^{\rho\sigma} - \frac{1}{6} F_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} + \frac{i}{2} F_{\mu\rho} \bar{\psi} \gamma^\rho D_\nu \psi \right. \right.$$

$$\left. - \frac{3i}{2} F_{\mu\rho} \bar{\psi} \gamma_\nu D^\rho \psi + \frac{5}{36} \varepsilon_{\mu\rho\sigma\tau} F^{\rho\sigma} \bar{\psi} \gamma_5 \gamma^\tau D_\nu \psi - \frac{1}{8} \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} \bar{\psi} \gamma_5 \gamma^\tau D_\tau \psi \right) \right)$$

$$+ \kappa_3 \left(-\frac{4i}{3} \varepsilon_{\mu\nu\rho\sigma} \bar{\psi} \gamma^\sigma D^\rho D^\tau D_\tau \psi + q \left(-\frac{16}{3} F_{\mu\rho} F_{\nu\sigma} F^{\rho\sigma} + \frac{4}{3} F_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} - \frac{10i}{3} F_{\mu\rho} \bar{\psi} \gamma^\rho D_\nu \psi \right. \right.$$

$$\left. - \frac{34i}{3} F_{\mu\rho} \bar{\psi} \gamma_\nu D^\rho \psi + \frac{20i}{3} F_{\mu\nu} \bar{\psi} \gamma^\rho D_\rho \psi - \frac{11}{3} \varepsilon_{\mu\rho\sigma\tau} F^{\rho\sigma} \bar{\psi} \gamma_5 \gamma^\tau D_\nu \psi + 2 \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} \bar{\psi} \gamma_5 \gamma^\tau D_\tau \psi \right) \right)$$

$$+ \kappa_4 \left(-\frac{7i}{6} \varepsilon_{\mu\nu\rho\sigma} \bar{\psi} \gamma^\sigma D^\rho D^\tau D_\tau \psi + q \left(-\frac{8}{3} F_{\mu\rho} F_{\nu\sigma} F^{\rho\sigma} + \frac{2}{3} F_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} - \frac{10i}{3} F_{\mu\rho} \bar{\psi} \gamma^\rho D_\nu \psi \right. \right.$$

$$\left. - \frac{13i}{3} F_{\mu\rho} \bar{\psi} \gamma_\nu D^\rho \psi + \frac{19i}{6} F_{\mu\nu} \bar{\psi} \gamma^\rho D_\rho \psi + \frac{2}{3} \varepsilon_{\mu\rho\sigma\tau} F^{\rho\sigma} \bar{\psi} \gamma_5 \gamma^\tau D_\nu \psi - \frac{1}{4} \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} \bar{\psi} \gamma_5 \gamma^\tau D_\tau \psi \right) \right)$$

$$+ \kappa_5 \left(\frac{i}{3} \varepsilon_{\mu\nu\rho\sigma} \bar{\psi} \gamma^\sigma D^\rho D^\tau D_\tau \psi + q \left(-\frac{4i}{3} F_{\mu\rho} \bar{\psi} \gamma^\rho D_\nu \psi + \frac{2i}{3} F_{\mu\rho} \bar{\psi} \gamma_\nu D^\rho \psi + \frac{5i}{3} F_{\mu\nu} \bar{\psi} \gamma^\rho D_\rho \psi \right. \right.$$

$$\left. + \frac{2}{3} \varepsilon_{\mu\rho\sigma\tau} F^{\rho\sigma} \bar{\psi} \gamma_5 \gamma^\tau D_\nu \psi - \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} \bar{\psi} \gamma_5 \gamma^\tau D_\tau \psi \right) \right)$$

$$+ \kappa_6 \left(2 q \varepsilon_{\mu\rho\sigma\tau} F^{\rho\sigma} \bar{\psi} \gamma_5 \gamma^\tau D_\nu \psi \right)$$

$$+ \kappa_7 \left(-\frac{4i}{3} \varepsilon_{\mu\nu\rho\sigma} \bar{\psi} \gamma^\sigma D^\rho D^\tau D_\tau \psi + q \left(\frac{4i}{3} F_{\mu\rho} \bar{\psi} \gamma^\rho D_\nu \psi + \frac{4i}{3} F_{\mu\rho} \bar{\psi} \gamma_\nu D^\rho \psi - \frac{8i}{3} F_{\mu\nu} \bar{\psi} \gamma^\rho D_\rho \psi \right. \right.$$

$$\left. \left. \left. \left. - \frac{2}{3} \varepsilon_{\mu\rho\sigma\tau} F^{\rho\sigma} \bar{\psi} \gamma_5 \gamma^\tau D_\nu \psi + 2 \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} \bar{\psi} \gamma_5 \gamma^\tau D_\tau \psi \right) \right) \right) \right) .$$

This result reduces to the one found before in [26] for $\kappa_i = 0$. One immediately notices that, apart from the usual commutative divergences and the 3-photon term $\mathcal{L}_{1,A}$, all other terms are electron-photon interactions: they have be transformed and expressed via $\mathcal{L}_i$.

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As usual, we write the one-loop divergent part as

$$\Gamma^{(1)}_{\text{div}} = -\mathcal{L}'_{\text{ct}}, \tag{36}$$

and add counterterms $\mathcal{L}'_{\text{ct}}$ to the classical action to cancel divergences after quantization. In this manner we obtain the bare lagrangian:

$$
\begin{aligned}
\mathcal{L}'_{\text{NC}} + \mathcal{L}'_{\text{ct}} \;=\; & -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\left(1 - \frac{4}{3}\frac{q^2}{(4\pi)^2\epsilon}\right) + \frac{1}{2}\left(i\bar\varphi\bar\sigma^\mu(D_\mu\varphi) + \text{h.c.}\right)\left(1 - 2\frac{q^2}{(4\pi)^2\epsilon}\right) \\
& + \frac{1}{2}q\mu^{\frac{\epsilon}{2}}\theta^{\mu\nu}\left(F_{\mu\rho}F_{\nu\sigma}F^{\rho\sigma} - \frac{1}{4}F_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right)\left(1 + \frac{q^2}{(4\pi)^2\epsilon}\frac{-4(1 + \kappa_2 - 8\kappa_3 - 4\kappa_4)}{3}\right) \\
& + \frac{1}{16}q\mu^{\frac{\epsilon}{2}}\theta^{\mu\nu}\Delta^{\alpha\beta\gamma}_{\mu\nu\rho}F_{\alpha\beta}\left(i\bar\varphi\bar\sigma^\rho(D_\gamma\varphi) + \text{h.c.}\right)\left(1 + \kappa_2 + \frac{q^2}{(4\pi)^2\epsilon}\frac{-5 + 3\kappa_2 - 20\kappa_3 - 20\kappa_4 - 8\kappa_5 + 8\kappa_7}{3}\right) \\
& + \left(i\theta^{\mu\nu}\varepsilon_{\mu\nu\rho\sigma}\bar\varphi\bar\sigma^\sigma D^\rho D^2\varphi + \text{h.c.}\right)\left(\kappa_3 + \frac{q^2}{(4\pi)^2\epsilon}\frac{-1 - \kappa_2 + 16\kappa_3 + 14\kappa_4 - 4\kappa_5 + 16\kappa_7}{12}\right) \\
& + q\mu^{\frac{\epsilon}{2}}\left(i\theta^{\mu\nu}F_{\mu\rho}\bar\varphi\bar\sigma_\nu D^\rho\varphi + \text{h.c.}\right)\left(\kappa_4 + \frac{q^2}{(4\pi)^2\epsilon}\frac{-1 + 9\kappa_2 + 68\kappa_3 + 26\kappa_4 - 4\kappa_5 - 8\kappa_7}{6}\right) \\
& + q\mu^{\frac{\epsilon}{2}}\left(i\theta^{\mu\nu}F_{\mu\nu}\bar\varphi\bar\sigma^\rho D_\rho\varphi + \text{h.c.}\right)\left(\kappa_5 + \frac{q^2}{(4\pi)^2\epsilon}\frac{-1 - \kappa_2 - 20\kappa_3 - 6\kappa_4 - 4\kappa_5 + 8\kappa_7}{4}\right) \\
& + q\mu^{\frac{\epsilon}{2}}\left(\theta^{\mu\nu}\varepsilon_{\mu\rho\sigma\tau}F^{\rho\sigma}\bar\varphi\bar\sigma^\tau D_\nu\varphi + \text{h.c.}\right)\left(\kappa_6 + \frac{q^2}{(4\pi)^2\epsilon}\frac{-6 - 5\kappa_2 + 132\kappa_3 - 24\kappa_4 - 24\kappa_5 - 72\kappa_6 + 24\kappa_7}{36}\right) \\
& + q\mu^{\frac{\epsilon}{2}}\left(\theta^{\mu\nu}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}\bar\varphi\bar\sigma^\tau D_\tau\varphi + \text{h.c.}\right)\left(\kappa_7 + \frac{q^2}{(4\pi)^2\epsilon}\frac{7 + \kappa_2 - 16\kappa_3 + 2\kappa_4 + 4\kappa_5 - 16\kappa_7}{8}\right).
\end{aligned}
\tag{37}
$$

The ϵ is regularization parameter from dimensional regularization. Now we can read off the values of the bare couplings and fields. From the commutative part we obtain known renormalizations

$$\varphi_0 \;=\; \sqrt{Z_2}\,\varphi = \sqrt{1 - 2\frac{q^2}{(4\pi)^2\epsilon}}\,\varphi, \tag{38}$$

$$A_0^\mu \;=\; \sqrt{Z_3}\,A^\mu = \sqrt{1 - \frac{4}{3}\frac{q^2}{(4\pi)^2\epsilon}}\,A^\mu, \tag{39}$$

$$
\begin{aligned}
q_0 \;=\; & \mu^{\frac{\epsilon}{2}}Z_3^{-1/2}Z_2^{-1}\left(1 - 2\frac{q^2}{(4\pi)^2\epsilon}\right)q \\
\;=\; & \mu^{\frac{\epsilon}{2}}\left(1 + \frac{2}{3}\frac{q^2}{(4\pi)^2\epsilon}\right)q.
\end{aligned}
\tag{40}
$$

The noncommutative part of divergences will give the bare couplings, $(\kappa_i)_0$. But we see that along with $\mathcal{L}_i$, 3-photon term $\mathcal{L}_{1,A}$ also gets quantum correction from the fermion loops, though its coefficient is in the classical lagrangian fixed to 1. This implies that a rescaling of noncommutativity parameter θ is necessary. The bare θ_0 is given by

$$\theta_0^{\mu\nu} = \left(1 - \frac{4}{3}\frac{q^2}{(4\pi)^2\epsilon}(\kappa_2 - 8\kappa_3 - 4\kappa_4)\right)\theta^{\mu\nu}. \tag{41}$$

Using (38-41) we obtain the running of the κ_i:

$$(\kappa_2)_0 = \kappa_2 + \frac{1}{3}\frac{q^2}{(4\pi)^2\epsilon}\Big(1 + 13\kappa_2 + 4\kappa_2(\kappa_2 - 8\kappa_3 - 4\kappa_4) - 52\kappa_3 - 36\kappa_4 - 8\kappa_5 + 8\kappa_7\Big) \tag{42}$$

$$(\kappa_3)_0 = \kappa_3 + \frac{1}{12}\frac{q^2}{(4\pi)^2\epsilon}\Big(-1 - \kappa_2 + 40\kappa_3 + 16\kappa_3(\kappa_2 - 8\kappa_3 - 4\kappa_4) - 4\kappa_5 + 16\kappa_7\Big) \tag{43}$$

$$(\kappa_4)_0 = \kappa_4 + \frac{1}{6}\frac{q^2}{(4\pi)^2\epsilon}\Big(-1 + 9\kappa_2 + 68\kappa_3 + 38\kappa_4 + 8\kappa_4(\kappa_2 - 8\kappa_3 - 4\kappa_4) - 4\kappa_5 - 8\kappa_7\Big) \tag{44}$$

$$(\kappa_5)_0 = \kappa_5 + \frac{1}{12}\frac{q^2}{(4\pi)^2\epsilon}\Big(-3 - 3\kappa_2 - 60\kappa_3 - 18\kappa_4 + 12\kappa_5 + 16\kappa_5(\kappa_2 - 8\kappa_3 - 4\kappa_4) + 24\kappa_7\Big) \tag{45}$$

$$(\kappa_6)_0 = \kappa_6 + \frac{1}{36}\frac{q^2}{(4\pi)^2\epsilon}\Big(-6 - 5\kappa_2 + 132\kappa_3 - 24\kappa_4 - 24\kappa_5 + 48\kappa_6(\kappa_2 - 8\kappa_3 - 4\kappa_4) + 24\kappa_7\Big) \tag{46}$$

$$(\kappa_7)_0 = \kappa_7 + \frac{1}{24}\frac{q^2}{(4\pi)^2\epsilon}\Big(21 + 3\kappa_2 - 48\kappa_3 + 6\kappa_4 + 12\kappa_5 + 32\kappa_7(\kappa_2 - 8\kappa_3 - 4\kappa_4)\Big). \tag{47}$$

A somewhat unusual quadratic running follows from renormalization of $\theta^{\mu\nu}$. In particular, we obtain

$$(\kappa_2)_0 - 4(\kappa_3)_0 - 2(\kappa_4)_0 = (1 + \frac{4}{3}\frac{q^2}{(4\pi)^2\epsilon})(\kappa_2 - 8\kappa_3 - 4\kappa_4)(\kappa_2 - 4\kappa_3 - 2\kappa_4) + \frac{1}{3}\frac{q^2}{(4\pi)^2\epsilon}(3 + 5\kappa_2 - 160\kappa_3 - 74\kappa_4).$$

We thus see that constraint (32) is not preserved for the bare couplings, that is, that coupling constants κ_2, κ_3 and κ_4 do not renormalize consistently with the SW expansion. Running of κ_5, κ_6 and κ_7 on the other hand obstructs (25-26) not.

V.   ANOMALY-SAFE THEORIES

The conclusion is therefore that the SW-expanded chiral electrodynamics (24) is not perturbatively renormalizable. Of course, this theory is not renormalizable for another, stronger reason: the existence of the chiral anomaly. It was namely shown in [31, 32] that in the θ-expanded theories, anomalies and anomaly-cancellation conditions are exactly the same as in the corresponding commutative theories, and know that chiral electrodynamics contains the chiral anomaly. So there is a natural the question: if we construct a model consisting of several fermions in different representations of noncommutative U(1) and impose the anomaly-cancellation conditions, would renormalizability improve as it happens for the GUT compatible models discussed in [27–29]? Unfortunately, as we shall see in the following, that this is not the case. Though the anomaly-cancellation conditions

$$\sum_i q_i = 0, \qquad \sum_i q_i^3 = 0 \tag{48}$$

remove the 3-photon vertex from the action and make room for an arbitrary renormalization of $\theta^{\mu\nu}$, this additional renormalization cannot improve overall renormalizability of the model.

To show this let us shortly discuss the classical action. We assume that we have a set of fermion fields $\hat\varphi_i$, $i = 1,\ldots N$ with electric charges q_i. It is known [33] that in the θ-expanded theories each noncommutative field comes with its own noncommutative potential $\hat A_i^\mu$; all of them however for $\theta = 0$ reduce to the same A^μ. The $\hat A_i^\mu$ are different because their corresponding SW maps differ: they depend on charges q_i. Therefore, in order to obtain the action with the

correct limit, in the sum

$$\mathcal{L}_{\mathrm{C}} = \sum_{i=1}^{N} \bar{\varphi}_i \bar{\sigma}_{\mu} (\partial^{\mu} + \mathrm{i} q_i A_i^{\mu}) \varphi_i - \frac{1}{4} \sum_{i=1}^{N} F_i^{\mu\nu} F_{i,\mu\nu} \tag{49}$$

we first need to rescale gauge fields $A_i^{\mu} \to \sqrt{c_i} A_i^{\mu} = \sqrt{c_i} A^{\mu}$ and charges $q_i \to \frac{1}{\sqrt{c_i}} q_i$. We then get

$$\mathcal{L}_{\mathrm{C}} = \sum_{i=1}^{N} \bar{\varphi}_i \bar{\sigma}^{\mu} (\partial_{\mu} + \mathrm{i} q_i A_{\mu}) \varphi_i - \frac{1}{4} \sum_{i=1}^{N} c_i F^{\mu\nu} F_{\mu\nu}. \tag{50}$$

To associate (50) with the lagrangian of the usual commutative theory we fix the sum of weights c_i to 1, $\sum_{i=1}^{N} c_i = 1$. In the noncommutative U(1) case we are free to choose $c_i = 1/N$; in other cases, for example in noncommutative generalizations of the Standard Model, analogous relationsare more complicated, [22]. The same rescaling applied to the noncommutative part of the lagrangian gives, for the boson vertex

$$\mathcal{L}_{1,A} = \frac{\sum_i q_i}{2N} \theta^{\mu\nu} \left(F_{\mu\rho} F_{\nu\sigma} F^{\rho\sigma} - \frac{1}{4} F_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right), \tag{51}$$

and clearly in the anomaly-safe model in which $\sum_i q_i = 0$ we obtain that this term vanishes, $\mathcal{L}_{1,A} = 0$. Fermion terms are on the other hand unchanged as they are mutually independent for each field:

$$\mathcal{L}_{2,\varphi_i} = \frac{\mathrm{i}}{16} \theta^{\mu\nu} \Delta_{\mu\nu\rho}^{\alpha\beta\gamma} F_{\alpha\beta} \sum_i q_i \bar{\varphi}_i \bar{\sigma}^{\rho} (\partial_{\gamma} + \mathrm{i} q_i A_{\gamma}) \varphi_i + \mathrm{h.c.} \tag{52}$$

$$\mathcal{L}_{3,\varphi_i} = \mathrm{i} \theta^{\mu\nu} \varepsilon_{\mu\nu\rho\sigma} \sum_i \bar{\varphi}_i \bar{\sigma}^{\sigma} D^{\rho} D^2 \varphi_i + \mathrm{h.c.}$$

$$\mathcal{L}_{4,\varphi_i} = \mathrm{i} \theta^{\mu\nu} F_{\mu\rho} \sum_i q_i \bar{\varphi}_i \bar{\sigma}_{\nu} D^{\rho} \varphi_i + \mathrm{h.c.}$$

$$\mathcal{L}_{5,\varphi_i} = \mathrm{i} \theta^{\mu\nu} F_{\mu\nu} \sum_i q_i \bar{\varphi}_i \bar{\sigma}^{\rho} D_{\rho} \varphi_i + \mathrm{h.c.}$$

$$\mathcal{L}_{6,\varphi_i} = \theta^{\mu\nu} \varepsilon_{\mu\rho\sigma\tau} F^{\rho\sigma} \sum_i q_i \bar{\varphi}_i \bar{\sigma}^{\tau} D_{\nu} \varphi_i + \mathrm{h.c.}$$

$$\mathcal{L}_{7,\varphi_i} = \theta^{\mu\nu} \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} \sum_i q_i \bar{\varphi}_i \bar{\sigma}^{\tau} D_{\tau} \varphi_i + \mathrm{h.c..}$$

We can extract the value of the one-loop divergences from our previous result either using the same rescaling of charges q_i by c_i or straightforwardly, by repeating the calculation. We obtain for the renormalized lagrangian:

$$\mathcal{L}'_{\text{NC}} + \mathcal{L}'_{\text{ct}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\left(1 - \frac{4}{3}\frac{\sum_i q_i^2}{(4\pi)^2\epsilon}\right) + \sum_i \left(\frac{\text{i}}{2}\bar{\varphi}_i\bar{\sigma}^\mu(D_\mu\varphi_i) + \text{h.c.}\right)\left(1 - 2\frac{q_i^2}{(4\pi)^2\epsilon}\right) \tag{53}$$

$$+\frac{1}{16}\mu^{\frac{\epsilon}{2}}\theta^{\mu\nu}\Delta_{\mu\nu\rho}^{\alpha\beta\gamma}F_{\alpha\beta}\sum_i(\text{i}q_i\bar{\varphi}_i\bar{\sigma}^\rho(D_\gamma\varphi_i) + \text{h.c.})\left(1 + \kappa_{i,2} + \alpha_{i,2}\frac{q_i^2}{(4\pi)^2\epsilon}\right)$$

$$+\theta^{\mu\nu}\varepsilon_{\mu\nu\rho\sigma}\sum_i(\text{i}\bar{\varphi}_i\bar{\sigma}^\sigma D^\rho D^2\varphi_i + \text{h.c.})\left(\kappa_{i,3} + \alpha_{i,3}\frac{q_i^2}{(4\pi)^2\epsilon}\right)$$

$$+\mu^{\frac{\epsilon}{2}}\theta^{\mu\nu}F_{\mu\rho}\sum_i(\text{i}q_i\bar{\varphi}_i\bar{\sigma}_\nu D^\rho\varphi_i + \text{h.c.})\left(\kappa_{i,4} + \alpha_{i,4}\frac{q_i^2}{(4\pi)^2\epsilon}\right)$$

$$+\mu^{\frac{\epsilon}{2}}\theta^{\mu\nu}F_{\mu\nu}\sum_i(\text{i}q_i\bar{\varphi}_i\bar{\sigma}^\rho D_\rho\varphi_i + \text{h.c.})\left(\kappa_{i,5} + \alpha_{i,5}\frac{q_i^2}{(4\pi)^2\epsilon}\right)$$

$$+\mu^{\frac{\epsilon}{2}}\theta^{\mu\nu}\varepsilon_{\mu\rho\sigma\tau}F^{\rho\sigma}\sum_i(q_i\bar{\varphi}_i\bar{\sigma}^\tau D_\nu\varphi_i + \text{h.c.})\left(\kappa_{i,6} + \alpha_{i,6}\frac{q_i^2}{(4\pi)^2\epsilon}\right)$$

$$+\mu^{\frac{\epsilon}{2}}\theta^{\mu\nu}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}\sum_i(q_i\bar{\varphi}_i\bar{\sigma}^\tau D_\tau\varphi_i + \text{h.c.})\left(\kappa_{i,7} + \alpha_{i,7}\frac{q_i^2}{(4\pi)^2\epsilon}\right),$$

with

$$\alpha_{i,2} = \frac{1}{3}(-5 + 3\kappa_{i,2} - 20\kappa_{i,3} - 20\kappa_{i,4} - 8\kappa_{i,5} + 8\kappa_{i,7}),$$

$$\alpha_{i,3} = \frac{1}{12}(-1 - \kappa_{i,2} + 16\kappa_{i,3} + 14\kappa_{i,4} - 4\kappa_{i,5} + 16\kappa_{i,7}),$$

$$\alpha_{i,4} = \frac{1}{6}(-1 + 9\kappa_{i,2} + 68\kappa_{i,3} + 26\kappa_{i,4} - 4\kappa_{i,5} - 8\kappa_{i,7}),$$

$$\alpha_{i,5} = \frac{1}{4}(-1 - \kappa_{i,2} - 20\kappa_{i,3} - 6\kappa_{i,4} - 4\kappa_{i,5} + 8\kappa_{i,7}),$$

$$\alpha_{i,6} = \frac{1}{36}(-6 - 5\kappa_{i,2} + 132\kappa_{i,3} - 24\kappa_{i,4} - 24\kappa_{i,5}$$
$$-72\kappa_{i,6} + 24\kappa_{i,7}),$$

$$\alpha_{i,7} = \frac{1}{8}(7 + \kappa_{i,2} - 16\kappa_{i,3} + 2\kappa_{i,4} + 4\kappa_{i,5} - 16\kappa_{i,7}).$$

Renormalization of fields and charges is the standard one,

$$\varphi_{i,0} = \sqrt{Z_{i,2}}\varphi_i = \sqrt{1 - 2\frac{q_i^2}{(4\pi)^2\epsilon}}\varphi_i,$$

$$A_0^\mu = \sqrt{Z_3}A^\mu = \sqrt{1 - \frac{4}{3}\frac{\sum_j q_j^2}{(4\pi)^2\epsilon}}A^\mu,$$

$$q_{i,0} = \mu^{\frac{\epsilon}{2}}Z_3^{-1/2}Z_{i,2}^{-1}\left(1 - 2\frac{q_i^2}{(4\pi)^2\epsilon}\right)q_i$$

$$= \mu^{\frac{\epsilon}{2}}\left(1 + \frac{2}{3}\frac{\sum_j q_j^2}{(4\pi)^2\epsilon}\right)q_i,$$

while noncommutativity $\theta^{\mu\nu}$ can renormalize arbitrarily. In order to try to use this fact we assume that it is of the form

$$\theta_0^{\mu\nu} = \left(1 + \alpha\frac{\sum_j q_j^2}{(4\pi)^2\epsilon}\right)\theta^{\mu\nu},$$

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with an arbitrary coefficient α which is to be determined from some renormalizability constraint. Renormalization of the κ_i follows:

$$(\kappa_{i,2})_0 = \kappa_{i,2} + \frac{1}{(4\pi)^2\epsilon}\left(-\alpha(1+\kappa_{i,2})\sum_j q_j^2 + \frac{1}{3}(1+9\kappa_{i,2}-20\kappa_{i,3}-20\kappa_{i,4}-8\kappa_{i,5}+8\kappa_{i,7})q_i^2\right),$$

$$(\kappa_{i,3})_0 = \kappa_{i,3} + \frac{1}{(4\pi)^2\epsilon}\left(-\alpha\kappa_{i,3}\sum_j q_j^2 + \frac{1}{12}(-1-\kappa_{i,2}+40\kappa_{i,3}+14\kappa_{i,4}-4\kappa_{i,5}+16\kappa_{i,7})q_i^2\right),$$

$$(\kappa_{i,4})_0 = \kappa_{i,4} + \frac{1}{(4\pi)^2\epsilon}\left(-\alpha\kappa_{i,4}\sum_j q_j^2 + \frac{1}{6}(-1+9\kappa_{i,2}+68\kappa_{i,3}+38\kappa_{i,4}-4\kappa_{i,5}-8\kappa_{i,7})q_i^2\right),$$

$$(\kappa_{i,5})_0 = \kappa_{i,5} + \frac{1}{(4\pi)^2\epsilon}\left(-\alpha\kappa_{i,5}\sum_j q_j^2 + \frac{1}{4}(-1-\kappa_{i,2}-20\kappa_{i,3}-6\kappa_{i,4}+4\kappa_{i,5}+8\kappa_{i,7})q_i^2\right),$$

$$(\kappa_{i,6})_0 = \kappa_{i,6} + \frac{1}{(4\pi)^2\epsilon}\left(-\alpha\kappa_{i,6}\sum_j q_j^2 + \frac{1}{36}(-6-5\kappa_{i,2}+132\kappa_{i,3}-24\kappa_{i,4}-24\kappa_{i,5}+24\kappa_{i,7})q_i^2\right),$$

$$(\kappa_{i,7})_0 = \kappa_{i,7} + \frac{1}{(4\pi)^2\epsilon}\left(-\alpha\kappa_{i,7}\sum_j q_j^2 + \frac{1}{8}(7+\kappa_{i,2}-16\kappa_{i,3}+2\kappa_{i,4}+4\kappa_{i,5})q_i^2\right).$$

But we easily observe that, however we fix α, expressions

$$(\kappa_{i,2})_0 - 4(\kappa_{i,3})_0 - 2(\kappa_{i,4})_0 = \left(1-\frac{1}{(4\pi)^2\epsilon}\alpha\sum_j q_j^2\right)(\kappa_{i,2}-4\kappa_{i,3}-2\kappa_{i,4}) - \frac{1}{(4\pi)^2\epsilon}\alpha\sum_j q_j^2$$

$$+\frac{1}{(4\pi)^2\epsilon}\frac{q_i}{3}(3+19\kappa_{i,2}-128\kappa_{i,3}-72\kappa_{i,4}),$$

cannot be zero.

VI. DISCUSSION

Before we discuss the meaning of our result let us consider briefly some limiting cases. The easiest case is when fermions are absent, $\varphi = 0$. We see that then there are no new noncommutative divergences and therefore no need to rescale $\theta^{\mu\nu}$, which is in accord with previously obtained behavior of the $SU(N)$ gauge theories, [20]. We discussed in [26] the minimal or 'little' case in which all $\kappa_i = 0$, $i = 2, \ldots, 7$, that is, in which complete noncommutative fermion correction reduces to $\mathcal{L}_2$. As it can be seen from (42-47), this one classical term generates after quantization all six $\mathcal{L}_i$; even the fermion propagation changes. This is in a way nice result, as it shows again a specific relation between the spatial and the gauge degrees of freedom: fermion propagation changes because of the photon loops. Quantum corrections generate noncommutative interactions even when they are absent from the classical lagrangian, that is for $1 + \kappa_2 = 0$, $\kappa_i = 0$, $i = 3, \ldots, 7$; this effect was discussed before in the case of Dirac fermions in [37].

Let us summarize the obtained result. We started classically with the most general action permitted by the Seiberg-Witten map. As the amount of nonuniqueness of the Seiberg-Witten expansion is huge, the initial lagrangian (24)

contains essentially all terms allowed by dimension and gauge covariance. Denote

$$\mathcal{L}'_{1,A} = \lambda_1 \theta^{\mu\nu} F_{\mu\rho} F_{\nu\sigma} F^{\rho\sigma} + \lambda_2 \theta^{\mu\nu} F_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}. \tag{54}$$

There are only two conditions in the lagrangian which distinguish the origin of the separate terms, that is which signify that the action was derived from (8) through the SW map: they are

$$\lambda_1 + 4\lambda_2 = 0, \qquad \kappa_2 - 4\kappa_3 - 2\kappa_4 = 0. \tag{55}$$

The first relation, the ratio between the two 3-photon terms, is stable under quantization; the second relation is broken after renormalization of the theory. This means that the SW map is not compatible with quantization: clearly, this happens only when fermions are present. This implies that the θ-expanded chiral electrodynamics is not renormalizable, and we are forced to conclude more generally, that the θ-expanded theories cannot be considered as fundamental or basic theories which provide representations of gauge symmetry on the Moyal space. They can probably give a good effective description of the effects of noncommutativity, but we expect that in a fundamental noncommutative gauge theory matter will have to be included in a different way.

Acknowledgments

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